DIFFERENTIAL GEOMETRY, EXAM ONE

MATH 3630/5630, "SPRING" 2019

Instructions: This is an individual effort exam. You may use your texts or course notes, but you may not search the internet or discuss this exam with anyone but me until after the deadline has passed. I encourage you to talk with me if you need to. You may use SageMath, Maple, Mathematica, or something similar to help you with tedious computation. You are not required to type up your solutions, but please make sure what you turn in is legible and well-organized.

Each question here is worth 15 points of course credit. There are twelve questions here. You probably won't [definitely won't] have time to do them all, so pick out a few and do them well. I think a good performance would be to do half of these really well.

This exam is due by 5pm on Friday. You may turn it in at the math department office, or by dropping it off at my office. (Go ahead and slip it under the door if I am not in.) Good Luck!

Task 1 (Parametric Curves)**.** This task has two parts:

(1) Write a short essay answering this question, using examples:

> Why do we study parametric curves, rather than curves cut out by equations? What do we gain by making this choice? What do we "lose?" That is, what new problems do we pick up? Why is the trade worth it?

(2) Consider a curve in $\mathbb{R}^3$ as described below. Find a parametrization for this curve.
Let C be the surface of the standard right circular cone:

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$$

Our curve spirals around C in such a way to both (1) pass through the point $(1, 0, 1)$, and (2) have tangent lines which make a constant angle with the vertical direction.

Task 2 (Reparametrization and Arclength)**.** This task has two parts.

(1) Write a short essay on this topic, using examples:

> Why is it that we need to consider reparametrizations of a curve? Why have we chosen arclength as the "correct" parametrization? What is arclength, and how do we go about using it? How practical is reparametrization by arclength?

(2) Reparametrize the curve below by arclength:

$$\alpha(t) = \left(e^t \cos(t), e^t \sin(t), e^t\right)$$

Task 3 (Regularity)**.** This task has two parts:

(1) There are several geometric reasons why we choose to work with only *regular* parametrized curves. Write a short essay to describe a few of those reasons, using examples to illustrate your points.
(2) Show that the curve $\beta(t) = (\sin(3t)\cos(t), \sin(3t)\sin(t), 0)$ is a regular curve. Then find the equation of its tangent line at the time $t = \pi/3$.

Task 4 (The Frenet-Serret Apparatus)**.** Let $\gamma(s)$ be a unit-speed curve with arclength parameter s. Use the standard notations for the Frenet-Serret apparatus of γ. This task has two parts:

(1) Show that $\kappa\tau = T' \cdot B'$, where the prime denotes a derivative with respect to s.
(2) Show that $\gamma' \cdot (\gamma'' \times \gamma''') = \kappa^2\tau$.

Task 5 (The Local Picture)**.** Suppose that $r(s)$ is a unit-speed parametrized space curve with $\kappa(s_0) \neq 0$. Draw a good sketch of $r(s)$ in a neighborhood of $r(s_0)$ and include information about the Frenet-Serret apparatus that helps understand the shape of the curve in this neighborhood.

Task 6 (Curvature)**.** Write a short essay that describes *two* different interpretations of the curvature κ of a space curve. Be sure to include as one of your descriptions a discussion of what κ means in terms of the motion of the Frenet-Serret framing of the curve.

Task 7 (Torsion)**.** Write a short essay that describes *two* different interpretations of the torsion τ of a space curve. Be sure to include as one of your descriptions a discussion of what τ means in terms of the motion of the Frenet-Serret framing of the curve.

Task 8 (Geometry I)**.** Let $\alpha(s)$ be a unit-speed parametrized curve. Show that the trace of α is a part of a straight line if, and only if, all of its tangent lines are parallel.

Task 9 (Geometry II)**.** Let $\alpha(s)$ be a unit-speed space curve with non-vanishing curvature. Prove that the trace of α lies in a plane if, and only if, there is a point P in $\mathbb{R}^3$ such that every osculating plane of α passes through P.

Task 10 (Geometry III)**.** Suppose that $\alpha(t)$ is a regular space curve. (It need not be unit-speed.) Suppose that there is a point O in $\mathbb{R}^3$ so that for all t, $\alpha(t) - O$ is orthogonal to the vector $T(t)$. Prove that the trace of α lies on some sphere.

Task 11 (Technical Prowess!)**.** Let $\alpha(s)$ be a unit-speed space curve with $\kappa > 0$ and $\tau > 0$. Define a new space curve β by

$$\beta(s) = \int_0^s B_\alpha(u)\ du,$$

where B_α is the binormal vector for α.

First, show that β is also a unit-speed curve. Then find the parts of the Frenet-Serret apparatus for β in terms of those for α.

Task 12 (Sufficiency of κ and τ)**.** Write a short essay on this topic, using examples to illustrate:

What are the *natural equations* of a space curve? Why is it that the natural equations are enough to tell us *everything* about that curve?