DIFFERENTIAL GEOMETRY, EXAM TWO

MATH 3630/5630, SPRING!! 2019

Instructions: This is an individual effort exam. You may use your texts or course notes, but you may not search the internet or discuss this exam with anyone but me until after the deadline has passed. I encourage you to talk with me if you need to. You may use SageMath, Maple, Mathematica, or something similar to help you with tedious computation. You are not required to type up your solutions, but please make sure what you turn in is legible and well-organized.

Each question here is worth 15 points of course credit. There are twelve questions here. You probably won't [definitely won't] have time to do them all, so pick out a few and do them well. I think a good performance would be to do half of these really well.

This exam is due by 4:30pm on Monday April 15. You may turn it in at the math department office, or by dropping it off at my office. (Go ahead and slip it under the door if I am not in.) Good Luck!

Task 1 (Parametric Surfaces)**.** This task has two parts:

(1) Write a short essay answering these questions, using examples:

> What is a parametrized surface? How does one describe a surface this way, and why is it useful? What are the trade-offs? That is, how does studying parametrized surfaces simplify things, and what complications do we have to deal with? What is *regularity* in this context, and why is it important?

(2) The shape of the Earth is really well modeled by a *prolate ellipsoid*, which is a surface cut out by an equation of the form

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

where a and b are suitably chosen constants. (Though they depend on the units of measurement used, a and b can be fairly large numbers.)

- Show how to parametrize some portion of a prolate ellipsoid as a graph.
- Show how to parametrize some portion of a prolate ellipsoid as a surface of revolution.

Task 2 (Curves in the plane)**.** Write a short essay on the basics of the **local** theory of curves in the plane $\mathbb{R}^2$. Be sure to discuss what parts of the theory are the same as for curves in space, and what is generally done differently.

Task 3 (Total Curvature)**.** Write an essay about the total curvature $\int_0^L \kappa(s)\,ds$ of a closed curve. Be sure to connect this global quantity to geometric properties of the curve. What happens for curves in the plane? What happens for curves in space?

Task 4 (The First Fundamental Form)**.** Write an essay on the theory behind the first fundamental form of a parametrized surface. What is it? How is it useful? What are some helpful ways to think about it?

Task 5 (The Second Fundamental Form)**.** Write an essay on the theory behind the second fundamental form of a parametrized surface. What is it? How is it useful? What are some helpful ways to think about it?

Task 6 (Euler's Curvature Idea)**.** Write a short essay that describes Euler's idea about measuring the curvature of a surface using planar slices. Be sure to include discussion of the idea, how one might compute something useful, and any relevant and important theorems.

Task 7 (Gauss' Curvature Idea)**.** Write a short essay that describes Gauss' idea about measuring the curvature of a surface using the Gauss map. Be sure to include discussion of the idea, how one might compute something useful, and any relevant and important theorems.

Task 8 (The Shape Operator)**.** What is the shape operator of a surface? How should one think about this important object? How is it computed? Write an essay encapsulating what you know.

Task 9 (The Hyperbolic Helicoid)**.** The hyperbolic helicoid is a very pretty parametrized surface described as follows:

$$\mathbf{x}(u,v) = \left(\frac{\sinh(v)\cos(u)}{1+\cosh(u)\cosh(v)}, \frac{\sinh(v)\sin(u)}{1+\cosh(u)\cosh(v)}, \frac{\cosh(v)\sinh(u)}{1+\cosh(u)\cosh(v)} \right)$$

(Okay, there is really a family of these surfaces, and I just chose one of them which I think will make computations simplest.) Compute some relevant quantities for this surface. Can you tell which points of this surface are elliptic, hyperbolic, *etc* for different values of the parameters u and v?

Task 10 (Graphs)**.** A generally useful class of parametrized surfaces consists of the graphs of functions $f : \mathbb{R}^2 \to \mathbb{R}$. Such a parametrization takes this form:

$$\mathbf{x}(u, v) = (u, v, f(u, v)).$$

Show how to compute some relevant quantities, and then answer these questions:

- What would have to be true of the function f for the corresponding graph surface to be a minimal surface?
- What would have to be true of the function f for the corresponding graph surface to be a Gaussian flat ($K = 0$) surface?

Task 11 (Surfaces of Revolution)**.** Another generally useful class of parametrized surfaces consists of the surfaces of revolution. To form one of these, one generally takes a parametrized curve $\gamma(t) = (x(t), z(t))$ in the xz-plane which never crosses the z-axis, and then spins the curve about that z-axis. The resulting parametrized surface has the form

$$\mathbf{x}(t, \theta) = (x(t)\cos(\theta), x(t)\sin(\theta), z(t)) .$$

Show how to compute the relevant quantities, and then answer these questions:

- Where are the principal directions are on a surface of revolution?
- Under what conditions on γ or its component functions can you guarantee that the Gaussian curvature of a surface of revolution is positive? negative? zero?

Task 12 (Ruled Surfaces)**.** A generally useful class of parametrized surfaces consists of the ruled surfaces. Such a parametrization takes this form:

$$\mathbf{x}(s, t) = \alpha(s) + t\beta(s),$$

where α and β are functions $\mathbb{R} \to \mathbb{R}^3$. We generally think of α as a curve in space parametrized by arclength and β as a vector field along the curve.

- When is such a surface regular?
- Find an expression for the Gaussian curvature of a ruled surface that depends only on the functions α, β, and their derivatives.
- Under what conditions will a ruled surface have Gaussian curvature equal to zero?