Stellar Models

1. Starting Point

In order to determine the characteristics of a star from the center to the surface, you need to solve the following formulae from the previous chapter simultaneously to determine values of $m$, $T$, $R$, $F$, $P$, for all points through the interior of the star.

Conservation of Mass formula
\[
\frac{dm}{dr} = 4\pi^2 \rho
\]

2-1

Thermal Equilibrium
\[
\frac{dF}{dr} = 4\pi^2 \rho q
\]

2-2

Hydrostatic Equilibrium
\[
\frac{dP}{dr} = -\rho \frac{GM}{r^2}
\]

2-3

Radiative Transfer
\[
\frac{dT}{dr} = -\frac{3\kappa \rho F}{4a c T^3 4\pi^2}
\]

2-17

These formula can also be written in terms of $dm$ rather than $dr$ (mass interval rather than radii intervals). As mentioned previously, sometimes a mass increment is able to provide better resolution of how the various characteristics of a star vary from the center to the surface.

Along with the above four formulae you have to have some auxiliary formulae

Opacity law (general form)
\[
\kappa = \kappa_i \rho^{aT^b}
\]

2-16

Energy Production (general form)
\[
q = q_0 \rho T^n
\]

2-18

And Equation of State
\[
P = P_{gas} + P_{rad}
\]

2-9, 2-11, 2-12, 2-13

where this could be based upon ideal gases or degenerate gases.

That means you have 7 formulae that need to be solved for the following unknown quantities – $m$, $T$, $R$, $F$, $P$. Can this be done? Certainly. But these formula have to be solved for each part of the star, at the same time – they must converge to common values.
for $m$, $T$, $R$, $F$, $P$ and they also have to be consistent with the other values that are involved, $\kappa$, $q$, $\rho$, which all depend upon, and influence the other values.

Typically the current method of solution is to use a grid system from the center of the star to the surface with defined values for $m$ or $r$ set up initially. To solve this grid of values, you need to have some boundary conditions – the conditions at the center and the surface. Boundary conditions are your starting and ending points, at least they are the values you want to end up with, and if you don’t then you may have made a mistake.

<table>
<thead>
<tr>
<th>Center (layer=0)</th>
<th>layer=1</th>
<th>layers = 2 → (n-1)</th>
<th>Surface (layer=n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>$T_1$</td>
<td>....</td>
<td>$T_n=T_{\text{eff}}$ or 0</td>
</tr>
<tr>
<td>$r_0(=0)$</td>
<td>$r_1$</td>
<td>....</td>
<td>$r_n=R$</td>
</tr>
<tr>
<td>$m_0(=0)$</td>
<td>$m_1$</td>
<td>....</td>
<td>$m_n=M$</td>
</tr>
<tr>
<td>$F_0 (=0)$</td>
<td>$F_1$</td>
<td>....</td>
<td>$F_n=L$</td>
</tr>
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<td>$P_0$</td>
<td>$P_1$</td>
<td>....</td>
<td>$P_n=0$</td>
</tr>
<tr>
<td>$\rho_0$, $\kappa_0$, $q_0$</td>
<td>$\rho_1$, $\kappa_1$, $q_1$</td>
<td>....</td>
<td>$\rho_n$, $\kappa_n$, $q_n=0$</td>
</tr>
<tr>
<td>$X_0$, $Y_0$, $Z_0$</td>
<td>$X_1$, $Y_1$, $Z_1$</td>
<td>....</td>
<td>$X_n$, $Y_n$, $Z_n$</td>
</tr>
</tbody>
</table>

At the center of the star you have the following conditions: $r=0$, $m=0$, $F=0$, but you should note that $T$, $P$, are not 0! So $T$, $P$ need initial values in the center. You may need to guess these values, or you could use approximations like equation 2-4. Also since the values of $T$, $P$ are not 0, values for density, opacity and energy generation are also needed. You also need to keep track of the values for the composition, $X$, $Y$, $Z$, which may not be uniform from the surface to the center. Typically this is something you decide on, but usually as a starting condition, you can assume there is a uniform composition throughout.

With the center conditions defined, you then go out a little ways from center to distance $r_1$, which has corresponding values for $m_1$, $F_1$, $T_1$, $P_1$, $\rho_1$, $\kappa_1$, $\mu_1$, $q_1$ – all of these must be calculated using the equations above. Since each of the main formulae are “difference” formulae, they can be considered as changes in value from one layer in the star to the next. So $dT$ can be viewed as $T_1-T_0$, $dr = r_1-r_0$, $dm = m_1-m_0$, $dP=P_1-P_0$, and so on. While technically this is an approximation of the differential equations, if the increments are small enough, the effect of using differences is not significant.

So you can solve the formula going from the center of the star to the surface, to the last layer of the star, often the layer=$n$. And like the center of the star, there are certain boundary conditions at the surface $r_n=R$, $m_n=M$, $F_n=L=4\pi R^2 \sigma T_{\text{eff}}^4$, $T_n=T_{\text{eff}}$ (or $T_n=0$), $P_n \approx 0$, $\rho_n \approx 0$. Again, some of these are simplifications but you have to have some sort of values to “aim for” as you go from the center outwards. Generally if you don’t get the values at the surface to match the values that come out of the various formulae, you may have to adjust things, such as the assumptions that you made about the central conditions, or you may have to change the overall conditions – perhaps you have wrong values for the total mass, luminosity and surface temperature?
Regardless of what you do, the computation of stellar models typically requires a lot of iterations over a grid with slight changes in values and the expected outcomes until the model converges to the solution that you are seeking. Obviously large, fast computers can help with this process and those are typically used in the computation of stellar models. Many of these models are available on the internet and in some cases you could even download a copy of the computer code to make your own models (assuming you have enough hard drive space to handle all of the codes).

2. Polytropes

What does one do if you don’t have a computer to solve all of this stuff? That was the situation before computers were widely available. In the “old” days people used to compute stellar models by hand. How is that possible? First of all you need to assume a few things to make the computations manageable –

1. Uniform composition throughout the interior. This is fundamentally wrong since higher density material should sink to the center. However, this will make the calculations of pressure, and opacity easy. This is sort of a way of saying that stars don’t change over time – so you’re basically ignoring the effects of evolution.
2. No complex motions are occurring inside of the star. This would usually be things like convection, or currents in the material that may arise due to the rotation of the star.
3. The center/surface characteristics are defined and set

The earliest simple models are known as polytropic models, which are models that tend to depend upon functions that have various exponents (powers). What do these functions entail? Look at formulas 2-1, 2-2, 2-3 and 2-17 – there are links between these formulae. These formulae would be easier to handle of we didn’t have a complex relationship between pressure, temperature and density – so let’s make that simpler. What if pressure were only a function of density and not temperature? If pressure and temperature were independent, then formula 2-1 and 2-3 can be solved by themselves since temperature doesn’t play into either of these or in any of the EOS formula (so this pretty much eliminates the use of radiation pressure).

If pressure is not a function of temperature, then the EOS formula is only a function of density or

\[ P = K\rho^\gamma \]  

And we define the value of \( \gamma \) as

\[ \gamma = 1 + \frac{1}{n} \]

And \( n \) is known as the polytropic index. For the situation of \( n=1.5 \), \( \gamma = 5/3 \), which is the index for a degenerate electron gas, while for \( n=3 \), \( \gamma=4/3 \) and that’s how a relativistic degenerate gas is defined. There are also situations where ideal gases can be described by this (I’ll get to this later).
Once this formula is defined for the density and pressure, it is possible to combine formulas for conservation of mass and hydrostatic equilibrium (2-1 and 2-3) and using some calculus you’d get the following messy formula

\[
\frac{(n+1)K}{4\pi Gn} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\rho}{\rho^{n-1} dr} \right) = -\rho
\]

Which is (believe it or not) rather simple – all you have here is a bunch of constants, density and radius. And if you know what the conditions are in the center and the surface \((r=0, R)\), then you can determine the value for the density throughout the star. And once that is defined, you can define the pressure, and then mass, then gravity.

But let’s keep messing around with things. Let’s assume that the density varies according to the relation

\[
\rho = \rho_c \theta^n
\]

Where \(\rho_c\)=central density, and \(\theta\) is now the variable that describes how the density changes, with \(0 \leq \theta \leq 1\), and \(n\)=polytropic index. Basically density now is a function of the central density (a constant) and varies as you go to the surface in a certain way defined by the polytropic index.

This formula can be put into 3-3 to make a new version of that function –

\[
\frac{(n+1)K}{4\pi Gn \rho_c^{n-1}} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta}{dr} \right) = -\theta^n
\]

Which looks really ugly, but it isn’t – the stuff in the [] is really just a constant of dimension length squared

\[
\left[ \frac{(n+1)K}{4\pi Gn \rho_c^{n-1}} \right] = \alpha^2
\]

We already have a way of defining distance in the star – the radius variable of \(r\). Instead of having another variable, we’ll combine \(r\) and \(\alpha\) into a new relation –

\(r=\alpha \xi\)

\(\xi\) is dimensionless (since \(r\) and \(\alpha\) have dimensions), and now we can put that into formula 3-4, and you get

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n
\]

Yes, it looks scary, but this is what is known as the Lane-Emden equation for a polytrope of index \(n\). Believe it or not, this is a really simple function to solve. Basically you define a value for \(n\) and you know what your limits are –

Center: \(\theta=1, \xi=0\), and \(d\theta/d\xi =0\)

And this goes to the surface, where \(\xi=\xi_1\) and \(R=\alpha \xi_1\) and also you know that the surface density is zero so \(\theta(\xi_1)=0\).
2.1 Using Polytropes

Equation 3-5 can be solved for a variety of function of \( n \), and these can be used to compare stellar structures under a variety of conditions. There are also various relations that can be derived from the polytrope relation that will help define certain characteristics of a star – short cuts believe it or not. Since the derivations of these things is rather messy and complex, I’m going to skip it and just give you the relationships.

One relation that can be found is the total mass of a polytropic star – which ends up being given by the following:

\[
M = -4\pi \alpha^3 \rho \frac{d\bar{\rho}}{d\bar{\xi}} \left(\frac{d\bar{\theta}}{d\bar{\xi}}\right)_{\delta_i}
\]

3-6

It is also possible to relate the central density to the average density and a polytropic constant \( D_n \) (which is only a function of \( n \)) -

\[
\rho_c = D_n \bar{\rho} = D_n \frac{M}{4\pi \frac{R^3}{3}}
\]

3-7

And there are also polytropic mass and radius constants \( (M_n \text{ and } R_n) \), which are related via the following –

\[
\left(\frac{GM}{M_n}\right)^{n-1} \left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}
\]

3-8

It should be emphasized that \( M_n \) and \( R_n \) are not actually the masses or radii of specific stars, but are functions that depend upon the polytropic index \( n \), and just provide adjustments to the values of the mass and radius (as in the above formula).

Equation 3-8 is actually quite revealing about certain physical conditions in stars, such as when \( n=3 \). In that case the mass is independent of radius and is dependent only on \( K \) and \( M_3 \). So what does that mean? You have to remember, these formulas come from the mass conservation formula and hydrostatic equilibrium (2-1, 2-3), so for various values of \( n \), (like 3) you have hydrostatic equilibrium (a stable star), and there are unique masses that can satisfy this condition for a given value of \( K \).

For \( n=1 \) there is another unique situation, now with the radius independent of mass and depending only on \( K \).

And the cases in between these two \( 1<n<3 \), you have a general relationship between mass and radius that is only a function of \( n \).

\[
R^{3-n} \propto \frac{1}{M^{n-1}}
\]

So radius decreases as the mass increases – the more massive a star is, the denser it is. This is exactly the conditions that define the characteristics of a degenerate star.
Now if you take the value of $K$ from equation 3-8 and put it back into its original formula of $P = K \rho^\gamma$ (3-1), you now have a relation for the central pressure, which after some manipulation becomes

$$P_c = (4\pi)^{1/3} B_n GM^{2/3} \rho_c^{4/3}$$  \hspace{1cm} 3-9

Where you have defined another polytropic constant, $B_n$.

Basically you can calculate how the pressure, mass, radius, density all vary from the inside to the outside of a star in an approximate way by using particular values for $n$ and known values for the constants. Here is a table of values for the polytropic constants mentioned here.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_n$</th>
<th>$M_n$</th>
<th>$R_n$</th>
<th>$B_n$</th>
</tr>
</thead>
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<td>1.0</td>
<td>3.290</td>
<td>3.14</td>
<td>3.14</td>
<td>0.233</td>
</tr>
<tr>
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<td>5.991</td>
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</tr>
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<td>0.157</td>
</tr>
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<td>1.89</td>
<td>9.54</td>
<td>0.145</td>
</tr>
<tr>
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<td>622.408</td>
<td>1.80</td>
<td>14.97</td>
<td>0.135</td>
</tr>
<tr>
<td>4.5</td>
<td>6189.47</td>
<td>1.74</td>
<td>31.84</td>
<td>0.115</td>
</tr>
</tbody>
</table>

### 2.2 Chandrasekhar Mass Limit

There are some very interesting polytropic models, in particular those that are like electron degenerate stars (equation 2-11). This happens when $n=1.5$. And now we can check out what this star is like without having to do all of the icky math. We can do this since we have polytropic relations to give us information about what is happening.

So for the case of $n=1.5$ we have from equation 3-8 the situation where the mass and radius are related by

$$R \propto M^{-1/3} = 1/M^{1/3}$$

And the average density goes as

$$\bar{\rho} \propto M / R^3 \propto M^2$$

So the density increases as the square of the mass! As the mass goes up and up, the density sky rockets until eventually it reaches the point of relativistic degenerate material ($n=3$). And this is that special situation for mass and $K$. Or put another way the mass will eventually equal a constant – this is the mass limit for degenerate objects, the Chandrasekhar mass limit for degenerate stars (white dwarfs).

When all is said and done, and you substitute in the values for the polytropic values we get that the mass for a degenerate object is simply given by

$$M_{Ch} = \frac{5.83}{\mu_c^2} \text{ solar masses}$$  \hspace{1cm} 3-10
While you may have learned that the Chandrasekhar mass limit had a specific value (1.4 \( M_\odot \)) there isn’t just one value since it will vary with composition (which changes the values for \( \mu_c \)).

What else can we figure out from polytropic models? Well the stuff above is based upon our combining of two of the main formula for stellar structures. Let’s see if we can combine the radiative transfer and hydrostatic equilibrium formulae (2-17 and 2-3).

\[
\frac{dT}{dr} = -\frac{3\kappa \rho F}{4a c T^3 4\pi r^2} \quad \text{and} \quad \frac{dP}{dr} = -\frac{\rho GM}{r^2}
\]

Let’s divide them into one another so that we get rid of the \( dr \) term. This gives us

\[
\frac{dP}{dT} = \frac{4a c T^3 \pi 4GM}{3\kappa F}
\]

Now let’s make it really simple by assuming all of the pressure is radiation pressure. This may seem extreme, but since we are looking at how radiation flows, it is only logical to look at how radiation impacts the pressure – also it is just easier to ignore the other pressure influences.

In that case we use equation 2-13

\[
P_{rad} = \frac{a}{3} T^4
\]

And we have to differentiate it to get

\[
\frac{dP_{rad}}{dT} = \frac{4}{3} a T^3
\]

Now we have two equations with \( dP/dT \), so let’s set them equal to get……

\[
F = L = \frac{4\pi c GM}{\kappa}
\]

3-11

\( F=L \) at the surface of the star (where \( \text{mass} = M \)).

So this equation basically says if you have all of your pressure from radiation, you have to balance the energy outflow (\( L \)) with gravity (\( GM \)) – remember, this came from the hydrostatic equilibrium formula to begin with so there must be stability in here somewhere.

This is the Eddington Luminosity formula which basically says that if a star has luminosity greater than \( L \) it will blow itself apart. This is not a common problem since gravity is usually so powerful it keeps the star in one piece. But stars should still have luminosity less than the value given by equation 3-11 to remain happy. If it doesn’t, it will experience mass loss.

3. Eddington’s Standard Model

Sir Arthur Eddington (1882 – 1944) is also well known for providing one of the first early stellar model (without a computer) which can be used to describe and predict various observed features for stars. Since he was a big shot in the game, this is also known as the “standard model”.

Notes 3 - 7
We have at the surface the values of mass \( M \) and energy flow \( L \). So somewhere inside the star we can assume at any location that there is a relation between the mass \( m \) and flux \( F \) that is related to these values at the surface, or put another way
\[
\frac{F}{m} = \eta \frac{L}{M}
\]
Where \( \eta \) is a variable that goes from the surface inwards. At the surface \( \eta = 1 \), and as you go into the star, \( \eta \) gets larger as \( m \) gets smaller. Since energy flow depends upon opacity, \( \kappa \), we need to consider how it changes as well. Generally opacity decreases as you go into a star. That means \( \kappa \) and \( \eta \) vary in the opposite sense as you go into/out of the star. Let’s assume that they are balanced so that
\[
\kappa \eta = \text{constant} = \kappa_s
\]
and \( \kappa_s \) gives us the surface opacity.

If you go back to how we got equation 3-11, it was by setting the pressure/temperature differences \( (dP/dT) \) equal to one another. What if they aren’t? Why should they be equal – one of the formula describes the overall pressure variation, while the other defines the radiation pressure variation. But we can compare those two \( dP/dT \) relations and divide them into one another to get
\[
\frac{dP_{rad}}{dP} = \frac{L}{4\pi GM} \kappa_s
\]
Notice that everything on the right is a constant; this basically becomes a relationship between the radiation pressure and the total pressure
\[
P_{rad} = \frac{L}{4\pi GM} \kappa_s P
\]
What does this tell us? That the radiation pressure and the total pressure have a constant ratio from the center to the surface - that they are proportional to one another throughout the star. If that is the case, we can make things easy by using a constant of proportionality between these two pressures.

If the total pressure \( = \) gas pressure + radiation pressure, then we can define a value \( \beta \) such that
\[
P_{rad} = (1 - \beta) P \quad \text{and} \quad P_{gas} = \beta P
\]
We use the value of \( \beta \) to measure the amount of radiation pressure relative to the overall pressure. If \( \beta = 0 \), all of the pressure is in the form of radiation pressure, and there is no gas pressure. If \( \beta = 1 \), then it is all gas pressure with no radiation pressure.

This can also be used with the Eddington luminosity since the luminosity approaches the Eddington value when \( \beta \) approaches 0 (when the pressure is all radiation pressure).

Let’s assume that the gas pressure is given by the ideal gas law. If that is the case we have

Notes 3 - 8
\[
P = \frac{P_{\text{rad}}}{(1 - \beta)} = \frac{P_{\text{gas}}}{\beta}
\]
\[
aT^4 = \frac{3\mu T}{3(1 - \beta) \beta \mu}
\]
\[
T = \left[ \frac{3\mu(1 - \beta)}{a\mu\beta} \right]^{1/3} \rho^{1/3}
\]

And if you put the \( T \) defined above back into the ideal gas law to get the EOS, you’ll end up with a polytrope \( P = K\rho^{4/3} \) which is just like equation 3-8 with \( n=3 \), so that there is a simple relationship between \( M \) and \( K \). Again, mass is only dependent on a bunch of constants – but in this case one of the constants is \( \beta \).

What does this particular mass equal?

Converting values to more convenient units we get
\[
M = \frac{1}{(\beta\mu)^2} \left( \frac{1 - \beta}{.003} \right)^{1/2} \]

where \( M \) is in solar masses. This relation is called *Eddington’s quartic equation*, and it is rather peculiar since it says that there is a range of values for masses of stars – between the range of \( \beta \) between 0 and 1. There are several important implications for this relation.

1. \( \beta \) decreases as \( M \) increases – radiation pressure is very important in high mass stars, not so major for low mass stars
2. We can use this relation combined with the Eddington relation to derive a relationship between stellar luminosity, \( \beta \), and mass, that has the form \( L \propto M^3 \) which is close to the relationship often used for Main Sequence stars!
3. As the star evolves and \( \mu \) changes, this will also change the influence of radiation pressure (\( \beta \)), such that as \( \mu \) goes up, \( \beta \) goes down. As more metals are produced (\( \mu \) goes up), the radiation pressure increases – this is especially true for supergiants and red giants.

This stuff is all rather inexact and really just provides a simple way of looking at how some large scale parameters change over time. These are not useful for detailed examinations of the physical characteristics of stars. Any realistic attempt at defining the changes of stars over time as stars evolve require full solutions of the equations that define stars, along with accurate values for the auxiliary relationships for \( \kappa \), \( \rho \) and \( q \).

### 4. Stellar Stability

So far we have only looked at situations where stars are stable, mainly because such situations are easy to deal with. But that situation is rather boring and in some cases quite inaccurate. In reality all stars have some level of instability, but how can you tell if it exists? And what forms of instability are there?

One test for instability is to basically “kick” a stellar model – if the star is stable, then the kick that you gave it would gradually die out. Typically this is done by altering some
physical value and seeing if the star can adjust its characteristics over time to remove this fluctuation. If it can’t and the fluctuation gets worse and worse, then you have an unstable star.

One of the simplest types of stability to check for is that of hydrostatic equilibrium. In most cases there is a balance between gas pressure and gravity. But what if there is a bunch of radiation pressure – a pressure that does not depend upon the density of material? This will change the contribution that gravity would normally have on the star, such that it is less than the case where there is no radiation pressure. This makes sense because the radiation pushes the star outwards and counteracts gravity.

But you still have the star in balance since a contraction of the star (increase in gravity) would cause the internal gas pressure to increase, the temperature to increase and the radiation pressure to increase – all of this would counteract the increase in gravity. And an expansion (decrease in gravity) would cause a decrease in the gas pressure, a decrease in the temperature and gravity which would have to work against less forces to keep the star together.

Another equilibrium to consider is that for energy, in this case the balance of energy production and loss, or all of the energy produced by nuclear fusion, \( L(\text{nuclear}) \), should balance with all of the energy that is given off, \( L(\text{emitted}) \). For a star to be happy these should be equal, \( L(\text{nuclear}) = L(\text{emitted}) \). But what if \( L(\text{nuclear}) > L(\text{emitted}) \)? What does this mean?

In the case where more energy is produced than is given off you’d have various side effects. There would be an increase in internal pressure, and temperature inside of the star. This would cause an expansion of the star. The expansion decreases the density, and as a result the energy generation would also decrease. So \( L(\text{nuclear}) \) gets smaller.

And going the other way, the situation where more energy is given off than is produced, \( L(\text{emitted}) > L(\text{nuclear}) \), also has an effect. In this case the star will have a decrease in its internal pressure and energy, and gravity takes advantage of that and is able to contract the star more. The result is that this heats things up, and compresses the material, and that gets the nuclear rate up, or \( L(\text{nuclear}) \) gets larger.

But do these things always happen as described above? No. Things are a bit more interesting when you have degenerate material. This is because degenerate material EOS has no temperature dependence. So what happens in this case?

If \( L(\text{nuclear}) > L(\text{emitted}) \), the pressure and density will change but the temperature wont since it isn’t linked to pressure and density! Temperature will instead be influenced by energy flow and since more energy is being produced than is given off, the temperature will increase. And when you increase the temperature you will increase the \( L(\text{nuclear}) \) rate – remember how strongly that depends on temperature? Basically for a degenerate material, as \( L(\text{nuclear}) \) gets large, it will just keep getting larger until it basically blows up. This leads to a thermonuclear run away event – this happens when there is violent,
rapid fusion in degenerate material, and it can cause what you’d likely describe as an explosion, such as is seen in a helium flash or a nova eruption. The resulting explosion can change things by altering the pressure and density of the material and remove the degeneracy – though that’s not a guarantee in all cases. Often computer models that get to these situations usually are not able to continue the calculations after this point. This is usually seen in the evolution calculations of low mass stars that undergo a helium flash. The evolution is usually only calculated up to the point of the shell flash, and then they try to approximate the evolution after that event.

5. Convection

Convection is the most common form of instability and in some stars it is the dominant form of energy transport in various parts of the star. While convection will not destroy a star, it can mess some stuff up. For simplicity it would be easiest if all energy were carried by the mechanism of radiative transfer (what is given in equation 2-17), however that isn’t entirely true, since convection can be very dominant in a star while the radiative processes are miniscule. Convection can also lead to the mixing of material, especially in cases where the convective flow penetrates into the star’s core and pulls up some of that material into other layers of the star. While convection involves the motion of material up and down, the overall result is that there isn’t a change in the mass distribution (the amount that goes up also goes down).

Convection doesn’t always happen, and in some stars it is seen in only certain areas. What determines whether convection does or does not occur? There is a fairly simple criteria to check for convection, something that Karl Schwarzschild developed in 1906.

Let’s say that you have a chunk of material, which has a pressure and density $P_1$, and $\rho_1$. If this chunk were to move up to a region of lower pressure (called $P_2$ and $\rho_2$), what should happen?

The pressure and density of the chunk of material should change, and they would now have values of $P^*$, $\rho^*$. What happens next? That depends upon how $P^*$, $\rho^*$ compare to values of $P_2$, $\rho_2$ – there are two situations.

If $\rho^* > \rho_2$ the chunk is too dense, and since denser material sinks, the chunk will go back down and there is no instability (no convection)

If $\rho^* < \rho_2$ the chunk is too low density compared to its surroundings, so it keeps going up. Now we have an instability, and in this case convection takes place.

But does the material move very far in the case of convection? That will depend upon how quickly the material can adjust to its surroundings. How quick is that? Go back to the discussion on the dynamical and thermal time scales (equations 2-20 and 2-21). In
general \( \tau_{\text{thermal}} \gg \tau_{\text{dyn}} \), so the time for material to heat up/cool down is longer than the time for motion. Therefore an object moves without changing temperature significantly or exchanging energy with its environment. Normally an object expands when it goes up, but if heat isn’t exchanged by the object with its environment its density doesn’t have to necessarily equal that of the area that it is located in. This would be an adiabatic expansion in which no energy is exchanged between the object and its environment.

For an adiabatic system we have

\[ P = K_a \rho^\gamma \]  (again)

Convection would occur if the following criteria is met, where the * represents the surrounding that the displaced material finds itself in –

\[ \left( \frac{dP}{d\rho} \right)_* \geq \left( \frac{dP}{d\rho} \right)_a \quad \text{or} \quad \gamma_* \geq \gamma_a \]

If \( \left( \frac{dP}{d\rho} \right)_* < \left( \frac{dP}{d\rho} \right)_a \) or \( \gamma_* < \gamma_a \) is the case, then there is no convection. Equation 3-13 is the Schwarzschild criteria for convection.

Basically the slope of the pressure relative to the density determines the likelihood of convection. The graph shown below is a situation of an object that was originally at location 1. The line marked “A” is the relationship between pressure and density. The lines marked “S” and “U” are two possible situations that may exist when the chunk of material moves to location 2. At location 2 there are three different values of density for the given value of pressure, and these are different because the slope of the pressure-density relation is different in each case. The densities at location 2 are all different, with “S” having the lowest density and “U” having the highest density.

In this example if the chunk is at location 2 and has a density along the “A” line, while the “U” and “S” lines mark the density/pressure relations of the material around the chunk. In the case of “S”, the density of the surroundings is lower than that of the chunk, so the chunk of material sinks back down. If the surroundings are represented by the “U” line at location 2, then the surroundings have a higher density than the chunk of material, which causes the material to keep rising. Therefore, S=stable against convection, while U=unstable and we have convection occurring. The slopes of each line is also different, with the “S” line having a shallower (lower) slope, and “U” having a larger slope. So if the material and the environment have different pressure/density gradients (slopes), convection may occur if the Schwarzschild criterion is met.
So when do the conditions in a star occur to cause convection? It would occur if the value of $\gamma_a$ were to decrease. This commonly happens when ionization occurs and can also occur when high opacity values are present. The high opacity will contribute to convection by influencing the radiation pressure which provides lift to material.

In many stars convection is often seen in regions where hydrogen is initially ionized. This causes a very rapid change in the value of $\gamma_a$ over a very short distance. In stars this usually happens in layers with $T=10,000$ - $20,000$ K. This is also a location where opacity values tend to peak.

Convection is also seen to occur in the cores of stars, usually near locations where the energy flow is large – usually only the most massive stars have convection in the core. When convection does happen it pretty much makes the radiative transfer laws no longer valid. They cannot accurately represent the temperature variation in the star. Unfortunately there is no simple way to describe convection, mainly because it is a 3-dimensional phenomenon. Often rough approximations are used to represent it.

In the case of an entirely adiabatic situation, you would have the following relation (rather than 2-17)

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma_a}\right) \frac{\mu GM}{R} \frac{1}{r^2}$$

The only problem is that in many cases you have radiative energy transport occurring as well as convection, which means the contribution of each to the energy transport has to be considered. And the relations that define how the temperature variation changes due to motion, requires knowledge about the distance that material is transported (what is known as the mixing-length), the velocity of the material, and the ability of the material to absorb energy. Generally there aren’t too many long-term evolutionary models that deal with convection in a rigorous manner, since convection involves short term motions. In stellar evolution models the effects of convection are usually approximated by simple relations rather than the actual convective motions. Generally these approximations vary from one model to the next since there isn’t a consistent way to quantify these effects over the long term.